MATH20132 Calculus of Several Variables. 2020-21

Problems 2: Continuity

The definition of continuity given in the notes *is that* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is continuous at* $\mathbf{a} \in U$ *if, and only if,* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$. *This has the* ε - δ *version*

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \forall \mathbf{x},\ |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \varepsilon.$$

1. Scalar-valued functions.

i. Let $1 \le i \le n$ and define the *i-th projection function* $p^i : \mathbb{R}^n \to \mathbb{R}$ by only retaining the i-th coordinate, so

$$p^i(\mathbf{x}) = p^i\left(\left(x^1, ..., x^n\right)^T\right) = x^i.$$

Verify the ε - δ definition to show that p^i is continuous on $\mathbb{R}^n$.

Remember, if $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$ *and* $|\mathbf{x} - \mathbf{a}| < \delta$ *then* $|x^i - a^i| < \delta$ *for each* $1 \le i \le n$.

A different proof of continuity was given in the lectures.

ii. Prove, by verifying the ε - δ definition that

$$f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto x^1 + x^2 + ... + x^n$$

is continuous on $\mathbb{R}^n$.

iii. Let $\mathbf{c} \in \mathbb{R}^n, \mathbf{c} \neq \mathbf{0}$, be a fixed vector. Prove, by verifying the ε - δ definition that $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$ is continuous on $\mathbb{R}^n$.

Hint Make use of the Cauchy-Schwarz inequality, $|\mathbf{c} \bullet \mathbf{d}| \le |\mathbf{c}|\,|\mathbf{d}|$ for $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$.

Part i is a special case of Part iii, with $\mathbf{c} = \mathbf{e}_i$, *while Part ii is the special case* $\mathbf{c} = (1, 1, ..., 1)^T$.

2 Prove, by verifying the ε - δ definition of continuity that the scalar-valued $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto xy$ is continuous on $\mathbb{R}^2$.

Hint If $\mathbf{a} = (a, b)^T \in \mathbb{R}^2$ is given write $f(\mathbf{x}) - f(\mathbf{a}) = xy - ab$ in terms of $x - a$ and $y - b$.

3 Prove, by verifying the ε - δ definition that the vector-valued function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix}$$

is continuous on $\mathbb{R}^2$.

Note For practice I have asked you to verify the definition, **not** to use any result that would allow you to look at each component separately.

4 Let $M_{m,n}(\mathbb{R})$ be the set of all $m \times n$ matrix of real numbers. Let $M \in M_{m,n}(\mathbb{R})$.

In the notes we showed that the function $\mathbf{x} \mapsto M\mathbf{x}$ is continuous on $\mathbb{R}^n$ by showing that each component function is continuous on $\mathbb{R}^n$. In this question we show it is continuous by verifying the ε - δ definition.

i. Prove that there exists $C > 0$, depending on M, such that $|M\mathbf{x}| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.

   Hint Write the matrix in row form as

$$M = \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^2 \\ \vdots \\ \mathbf{r}^m \end{pmatrix} \quad \text{when} \quad M\mathbf{x} = \begin{pmatrix} \mathbf{r}^1 \bullet \mathbf{x} \\ \mathbf{r}^2 \bullet \mathbf{x} \\ \vdots \\ \mathbf{r}^m \bullet \mathbf{x} \end{pmatrix}.$$

   What is $|M\mathbf{x}|$? Apply Cauchy-Schwarz to each $|\mathbf{r}^i \bullet \mathbf{x}|$.

ii. Deduce, by verifying the ε - δ definition, that the vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$ is continuous on $\mathbb{R}^n$.

5. Determine where each of the following maps $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. For $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$,

i.

$$f(\mathbf{x}) = \begin{cases} x + y & \text{if } y > 0 \\ x - y - 1 & \text{if } y \leq 0 \end{cases}$$

ii.

$$f(\mathbf{x}) = \begin{cases} x + y & \text{if } y > 0 \\ x - y & \text{if } y \leq 0 \end{cases}$$

Hint: Your arguments should split into three cases, $y > 0$, $y < 0$ and $y = 0$. You should make use of the fact that polynomials in x and y are continuous in open subsets of $\mathbb{R}^2$.

6. Return to the function of Question 10 Sheet 1, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{(x^2 - y)^2}{x^4 + y^2} \quad \text{for } \mathbf{x} = (x, y)^T \neq \mathbf{0} \quad \text{and} \quad f(\mathbf{0}) = 1.$$

i. Show that f is continuous at the origin along any straight line through the origin.

ii. Show that f is not continuous at the origin.

This is then an illustration of

$$\boxed{\forall \mathbf{v},\ \lim_{t \to 0} f(\mathbf{a} + t\mathbf{v}) = f(\mathbf{a}) \ \not\Longrightarrow\ \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).}$$

Linear Functions.

7. **Linear functions** The definition of a linear function $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is that

$$\mathbf{L}(\mathbf{u} + \mathbf{v}) = \mathbf{L}(\mathbf{u}) + \mathbf{L}(\mathbf{v}) \quad \text{and} \quad \mathbf{L}(\lambda \mathbf{u}) = \boldsymbol{\lambda} \mathbf{L}(\mathbf{u}),$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$.

i. Given $\mathbf{a} \in \mathbb{R}^n$ prove that $L : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \mathbf{a} \bullet \mathbf{x}$ is a linear function.

   This was stated without proof in the lectures.

ii. An example of Part i is, if $\mathbf{a} = (2, -5)^T \in \mathbb{R}^2$, then $f(\mathbf{x}) = \mathbf{a} \bullet \mathbf{x} = 2x - 5y$ is a linear function on $\mathbb{R}^2$. Show that

   a. $f(\mathbf{x}) = 2x - 5y + 2$ is not a linear function on $\mathbb{R}^2$,

   b. $f(\mathbf{x}) = 2x - 5y + 3xy$ is not a linear function on $\mathbb{R}^2$.

iii. Given $M \in M_{m,n}(\mathbb{R})$, an $m \times n$ matrix with real entries, prove that $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$ is a linear function.

   This was stated without proof in the lectures.

iv. Let $\mathbf{L} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\mathbf{L}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x + 2y \\ x - y + 1 \\ 5x \end{pmatrix}.$$

Show that $\mathbf{L}$ is not a linear function.

8. If $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function prove that there exists $C > 0$, depending on $\mathbf{L}$, such that

$$|\mathbf{L}(\mathbf{x})| \leq C\,|\mathbf{x}| \tag{1}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Deduce that $\mathbf{L}$ satisfies the ε - δ definition of continuous on $\mathbb{R}^n$.

Hint Apply a result from the lectures along with Question 4 above.

Additional Questions 2

9. Verify the ε - δ definition of continuity and show that the scalar-valued $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto x^2 y$ is continuous on $\mathbb{R}^2$.

Hint Given $\mathbf{a} = (a, b)^T \in \mathbb{R}^2$ write $x^2 y - a^2 b$ in terms of $x - a$ and $y - b$.

10. Let $1 \leq i \leq n$ and define $\rho^i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by omitting the i-th coordinate, so

$$\rho^i \left(\left(x^1, ..., x^n \right)^T \right) = \left(x^1, ..., x^{i-1}, x^{i+1}, ..., x^n \right)^T .$$

i. Verify the ε - δ definition of continuity and show that ρ^i is continuous on $\mathbb{R}^n$.

ii. For each $1 \leq i \leq n$ find $M_i \in M_{n-1,n}(\mathbb{R})$ such that $\rho^i(\mathbf{x}) = M_i \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. (Thus continuity follows from Question 4. We could, though, note that ρ^i is linear in which case continuity follows from Question 8.)